troids and axoids (*Liouville's Journal*, XVI, 9–129; 286–336). The application of these methods to the study of Mechanisms was first made systematically by Willis (1841); his processes were improved by Rankine (*Machinery and Millwork*, 1871); and the whole subject was reformed and freshly stated by Reuleaux (*Kinematik*, 1874), and has since been still further developed by Grashof and others. Peculiar interest attaches to the "tram-motion" from its furnishing to Leonardo da Vinci his famous discovery of the elliptic chuck for turning ovals on the lathe (Chasles, *Aperçu* 531).—*W. M. T.*]

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.*

By MR. MARCUS BAKER, Washington, D. C.

In April, 1883, Mr. James Main, formerly of the U. S. Coast and Geodetic Survey, published in the *Mathematical Magazine* a collection of forty-six (46) expressions for the area of a plane triangle, prefacing it with the remark that this collection "may be regarded as a matter of curiosity," and that about one-half of the formulæ are well known.

In the following August M. Ed. Lucas reprinted this collection in *Mathesis* in a classified form, separating the formulæ into five groups and adding one formula not contained in Mr. Main's list. The collection has also been reprinted in the third number of the *Tidsskrift for Mathematik*, 1883. Some two or three additional formulæ have since been printed in various mathematical publications.

The terms in which the area is expressed in Mr. Main's collection are angles, sides, perpendiculars, and radii of inscribed, escribed, and circumscribed circles. No formulæ are given involving medians or bisectors. In numbering Mr. Main has not counted those formulæ as distinct which arise from merely permuting the letters, nor has he in *every* case given all the forms possible to be obtained by permuting the letters, though he has generally done so. As numbered, then, he counts forty-six formulæ, but if every form be counted as a distinct one the total number is ninety-four.

M. Lucas, by making all possible permutations and adding one new form, makes the number 139, to which some two or three have been added since.

As the matter has proved of interest, the following collection has been made, which is a still further extension; the additional formulæ being chiefly due to introducing the medians and bisectors, not used in the former collections. In this collection Mr. Main's mode of numbering has been followed and formulæ

*Read before the Mathematical section of the Philosophical Society of Washington, January 7, 1885.

derived by permutation are not enumerated as distinct formulæ. Moreover, formulæ expressed in the same terms, but in different form, are also considered as but one. For example, in the former lists we find

$$\Delta = 2R^2 \sin A \sin B \sin C,$$

$$\Delta = \tfrac{1}{2}R^2 (\sin 2A + \sin 2B + \sin 2C),$$

and $$\Delta = \tfrac{2}{3}R^2 [\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B)]$$

given as three distinct formulæ whereas they are here counted as *one*, and the principle involved herein is employed throughout.

The total number of formulæ for the area of a plane triangle in this collection is ninety-three, not counting those arising from permutation. If these be counted as distinct the total number is two hundred and sixty-nine.

Owing to pressure of other duties and consequent lack of time some of the groups in this collection are not so fully worked up as had been planned.

It may be noted that a number of curious and interesting theorems may be obtained by equating different expressions for the area and reducing. In this collection we have classified, for convenience, all expressions for the area into five groups and each group into two parts.

Group I contains formulæ which M. Lucas has called *unique*, i. e. formulæ which do not admit of other similar formulæ by merely permuting the letters. In such formulæ *all* the sides, *all* the perpendiculars, *all* the medians, etc. must enter if one enters.

Group II contains formulæ which admit of *two* similar expressions by permutation giving three of a kind.

Group III contains formulæ which admit of *three* similar expressions by permutation giving four of a kind.

Group IV contains formulæ which admit of *five* similar expressions by permutation giving six of a kind.

Group V contains formulæ which admit of *eleven* similar expressions by permutation giving twelve of a kind.

Each group is divided into *two* parts, the *first containing* and the *second not containing* trigonometrical functions. In addition to the foregoing a group of miscellaneous expressions, not falling within the classification used, has been added, and called the Miscellaneous Group.

The notation used is as follows:

$\Delta =$ the area of the triangle;

$A, B, C =$ the angles;

$a, b, c =$ the sides opposite A, B, and C respectively

$s =$ semi perimeter $= \frac{1}{2}(a + b + c)$;

$R, r, r_a, r_b, r_c =$ the radii of circumscribed, inscribed, escribed circles respectively;
$h_a, h_b, h_c =$ the perpendiculars from A, B, and C respectively;
$m_a, m_b, m_c =$ the medians from A, B, and C respectively;
$\beta_a, \beta_b, \beta_c =$ the bisectors (internal) of the angles A, B, and C respectively; and
$\sigma = \frac{1}{2}(m_a + m_b + m_c)$.

GROUP I. PART I.

$\Delta =$

1. $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$

2. $\frac{4}{3}\sqrt{\sigma(\sigma - m_a)(\sigma - m_b)(\sigma - m_c)}$
$= \frac{1}{3}\sqrt{2m_a^2m_b^2 + 2m_b^2m_c^2 + 2m_c^2m_a^2 - m_a^4 - m_b^4 - m_c^4}$

3. $1 \div$
$$\sqrt{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)}$$
$= h_a^2h_b^2h_c^2 \div$
$$\sqrt{(h_ah_b+h_bh_c+h_ch_a)(-h_ah_b+h_bh_c+h_ch_a)(h_ah_b-h_bh_c+h_ch_a)(h_ah_b+h_bh_c-h_ch_a)}$$

4. $\sqrt{rr_ar_br_c}$

5. $\dfrac{abc}{4R}$

6. $\dfrac{2}{27R}\sqrt{(2m_a^2 + 2m_b^2 - m_c^2)(2m_b^2 + 2m_c^2 - m_a^2)(2m_c^2 + 2m_a^2 - m_b^2)}$

7. $\sqrt{\frac{1}{2}Rh_ah_bh_c}$

8. $\dfrac{r}{2}\sqrt{\dfrac{(r_a + r_b)(r_b + r_c)(r_c + r_a)}{R}}$

9. $\frac{1}{2}\sqrt[3]{abch_ah_bh_c}$

10. $\dfrac{abc}{r_a + r_b + r_c - r} = \dfrac{r^2(r_a + r_b)(r_b + r_c)(r_c + r_a)}{abc} = \left\{\dfrac{a^2 + b^2 + c^2}{\dfrac{1}{r^2} + \dfrac{1}{r_a^2} + \dfrac{1}{r_b^2} + \dfrac{1}{r_c^2}}\right\}^{\frac{1}{2}}$

11. $\sqrt{\frac{1}{3}(m_a^2 + m_b^2 + m_c^2) \div \left(\dfrac{1}{h_a^2} + \dfrac{1}{h_b^2} + \dfrac{1}{h_c^2}\right)}$
$= \dfrac{1}{3h_ah_bh_c}\sqrt{3(m_a^2 + m_b^2 + m_c^2)(h_a^2h_b^2 + h_b^2h_c^2 + h_c^2h_a^2)}$

12. $2R^2\dfrac{h_ah_bh_c}{abc} = \frac{1}{2}\dfrac{R}{s}(h_ah_b + h_bh_c + h_ch_a)$

GROUP I. PART I.—*Continued.*

$\Delta =$

13. $$\left(\frac{R^2}{12}(m_a^2 + m_b^2 + m_c^2)(h_a^2h_b^2 + h_b^2h_c^2 + h_c^2h_a^2)\right)^{\frac{1}{6}}$$

14. $$\frac{\beta_a\beta_b\beta_c}{s} \cdot \frac{a+b}{2c} \cdot \frac{b+c}{2a} \cdot \frac{c+a}{2b} = \tfrac{1}{4}\beta_a\beta_b\beta_c\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{a+b+c}\right)$$

15. $$\tfrac{3}{8}\beta_a\beta_b\beta_c\left[\frac{1}{\sqrt{2m_a^2 + 2m_b^2 - m_c^2}} + \frac{1}{\sqrt{2m_b^2 + 2m_c^2 - m_a^2}} + \frac{1}{\sqrt{2m_c^2 + 2m_a^2 - m_b^2}} - \frac{1}{\sqrt{2m_a^2 + 2m_b^2 - m_c^2} + \sqrt{2m_b^2 + 2m_c^2 - m_a^2} + \sqrt{2m_c^2 + 2m_a^2 - m_b^2}}\right]$$

16. $$\tfrac{1}{2}\sqrt{\tfrac{1}{2}\beta_a\beta_b\beta_c\left\{h_a + h_b + h_c - \frac{1}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}}\right\}}$$

17. $$\tfrac{1}{2}\sqrt{\beta_a\beta_b\beta_c\left\{\frac{1}{\frac{1}{r_a} + \frac{1}{r_b}} + \frac{1}{\frac{1}{r_b} + \frac{1}{r_c}} + \frac{1}{\frac{1}{r_c} + \frac{1}{r_a}} - \frac{1}{r}\right\}}$$

18. $$\sqrt{\tfrac{1}{8}r\beta_a\beta_b\beta_c\left[(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 1\right]}$$

19. $$\tfrac{1}{4}\sqrt{\frac{\beta_a\beta_b\beta_c}{R}\,\frac{(a+b)(b+c)(c+a)}{a+b+c}}$$

20. $$\frac{Rr}{\beta_a\beta_b\beta_c}\left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{c} + \frac{1}{a}\right)$$

21. $$\frac{h_ah_bh_c}{8abc}(r_a + r_b + r_c - r)^2.$$

GROUP I. PART II.

22. $$2R^2\sin A\sin B\sin C$$
$$= \tfrac{1}{2}R^2(\sin 2A + \sin 2B + \sin 2C)$$
$$= \tfrac{2}{3}R^2[\sin^3 A\cos(B - C) + \sin^3 B\cos(C - A) + \sin^3 C\cos(A - B)]$$

23. $$\tfrac{1}{2}R(a\cos A + b\cos B + c\cos C)$$
$$= R[a\cos C\cos A + b\cos A\cos B + c\cos B\cos C]$$

24. $$\tfrac{1}{4}(a^2\cot A + b^2\cot B + c^2\cot C) = \tfrac{1}{4}\cdot\frac{a^2 + b^2 + c^2}{\cot A + \cot B + \cot C}$$

GROUP I. PART II.—*Continued.*

$\Delta =$

25.* $\frac{1}{3}\dfrac{m_a^2 + m_b^2 + m_c^2}{\cot A + \cot B + \cot C}$

26. $\sqrt{2R\,{}_\prime\beta_a\,{}_\prime\beta_b\,{}_\prime\beta_c\,[1 + \cos(A - B) + \cos(B - C) + \cos(C - A)]}$

$= 2\sqrt{R\,{}_\prime\beta_a\,{}_\prime\beta_b\,{}_\prime\beta_c \sin(A + \frac{1}{2}B)\sin(B + \frac{1}{2}C)\sin(C + \frac{1}{2}A)}$

27. $R\sqrt[3]{h_a h_b h_c \sin A \sin B \sin C}$

28. $\frac{1}{3}R \sin A \sin B \sin C\left(\dfrac{ab}{h_c} + \dfrac{bc}{h_a} + \dfrac{ca}{h_b}\right)$

29. $\dfrac{h_a + h_b + h_c}{2\left(\dfrac{\cos\frac{1}{2}A}{{}_\prime\beta_a} + \dfrac{\cos\frac{1}{2}B}{{}_\prime\beta_b} + \dfrac{\cos\frac{1}{2}C}{{}_\prime\beta_c}\right)}$

30. $\dfrac{{}_\prime\beta_a \sin(C + \frac{1}{2}A) + {}_\prime\beta_b \sin(A + \frac{1}{2}B) + {}_\prime\beta_c \sin(B + \frac{1}{2}C)}{2\left(\dfrac{\cos\frac{1}{2}A}{{}_\prime\beta_a} + \dfrac{\cos\frac{1}{2}B}{{}_\prime\beta_b} + \dfrac{\cos\frac{1}{2}C}{{}_\prime\beta_c}\right)}$

31. $\dfrac{{}_\prime\beta_a\,{}_\prime\beta_b\,{}_\prime\beta_c\sqrt{(m_a^2 - h_a^2)(m_b^2 - h_b^2)(m_c^2 - h_c^2)}}{s(a - b)(b - c)(c - a)}.$

[TO BE CONTINUED].

DEMONSTRATION OF DESCARTES'S THEOREM AND EULER'S THEOREM.

By PROF. G. B. HALSTED, Austin, Texas.

DESCARTES'S THEOREM.

Cutting by diagonals the faces not triangles into triangles, the whole surface of any polyhedron contains a number of triangular faces four less than double the number of summits.

Proof.

For, joining all the summits by a single closed broken line, this cuts the surface into two skew polygons, each of which contains $S - 2$ triangles, where S is the number of summits.

* *Tidsskrift for Mathematik,* 8° Copenhagen, 1883, fifth series, first year, No. 4, p. 136.